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SON OF THE LINEAR SEARCH PROBLEM

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ABSTRACT

I wish to find something which is located on a certain road. I start at a point on the road, but I do not know in which direction the object sought is to be found. Somehow, I must incorporate in my way of searching the possibility that it is either to the right or to the left. Thus, I must search first to the right, and then to the left, and then to the right again until it is found. What is a good way of conducting this search, and what is a bad way?

This general problem can be phrased in many ways mathematically, some of which are answered in the papers in the bibliography. In this paper, we consider three well-known assumptions concerning the *a priori* guesses for the probability distribution on where the object is located. These concern uniform distribution on an interval, triangular distribution around the original point, and normal distribution about that point. The uniform distribution has a simple answer. For the triangular distribution, we obtain qualitative results and calculate approximate values for the turning points.

1. Introduction

The linear search problem concerns an object which is the subject of a search. The object is located on the real line **R** with a known or unknown probability distribution F. The searcher begins at 0 and moves continuously until finding the object. For each possible way of searching, there is a minimum distance traversed to reach each point of **R**. Each point not reached is assigned a path distance of ∞. The cost of reaching each point is some known function of the point and the path length, and the aim of the searcher is to minimize the expected cost with respect to the (known or unknown) distribution.

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The senior author introduced the problem in 1963 by private communication; at the same time it was independently proposed by Richard Bellman in the SIAM Review. The first publications were by the senior author [1] and Wallace Franck [5] who independently developed criteria, each of which purported to be necessary and sufficient conditions that a distribution F admit a minimizing search strategy of a particularly simple kind.[†] This search strategy consists of starting from 0 in one direction a positive distance during which the object is found, or at the end of which the searcher reverses his direction, searching on the other side of 0 a positive distance until either the object is found or direction once more reversed. The strategy is defined by the *turning points*, and for such distributions, we classify such sequences $\{x_n \mid n = 1,2,\dots\}$ as search strategies. For each such sequence, the cost of searching is taken to be the expected path length. To get this, we observe that for t between 0 and x_1, the path length is $|t|$. Between 0 and x_2, it is $|t| + 2|x_1|$, and in general, for $n \geqq 1$ and t between x_n and x_{n+2}, it is $|t| + 2|x_1| + \cdots + 2|x_{n+1}|$. Integrating, and grouping the terms which are multiples of $|t|$ and the various $|x_n|$, we get $M_1(F) + 2\Delta(\{x_n\}, F)$, where

$$M_1(F) = \int_{-\infty}^{\infty} |x|\, dF(x) \quad \text{and} \quad \Delta(\{x_n\}, F) = \sum_{n=1}^{\infty} |x_n|(1 - |F(x_n) - F(x_{n-1})|).$$

In this paper, we discuss three particular distributions, all of which meet the conditions both of Beck and of Franck. These are the uniform, triangular, and normal distributions. The results presented here concerning the uniform and triangular distributions have been known to the senior author since 1964, but not previously published. The normal distribution has long been the subject of some theoretical analysis. It has long been conjectured that for the minimizing sequence $\{x_n\}$, $|x_n| - |x_{n-1}| \to 0$ as $n \to \infty$. This is proved in this paper. Indeed, we show that $|x_n|$ is asymptotically $\sqrt{2n \ln n}$. The search has also been subject to numerical analysis, with mixed results [6]. We provide another numerical solution.

[†] The conditions described in the two articles are not equivalent, as can be seen by examining the function

$$\int_0^x |t|^{-1/4} |\sin(t)|^{|1/t|}\, dt,$$

which satisfies the condition in [1], but not that in [5]. Readers are invited to decide for themselves which condition, if either, is correct. Please note the significant erratum in [1] corrected at the end of [2].

2. Uniform and triangular distributions

The case of the uniform distribution is especially easy. Let $a<0<b$ and $F(x)=(x-a)/(b-a)$, $\forall x \in (a,b)$. For every search strategy $\{x_n\}$ with $0<x_1<x_3 \leqq b$, we define $\{y_n\}$ by omitting x_1, with $y_n = x_{n+1}$, $\forall n \geqq 1$. Then

$$\sum_n |x_n|(1-|F(x_n)-F(x_{n-1})|)-\sum_n |y_n|(1-|F(y_n)-F(y_{n-1})|)$$
$$=x_1((b-a)-(x_1-x_2))/(b-a)>0,$$

so that $0<x_1<x_3 \leqq b$ assures that $\{x_n\}$ is not a minimal search strategy. Similarly if $a \leqq x_3 < x_1 < 0$. Thus, $x_1 = a$ or b, with x_2 being the other. For these two strategies, the expected path lengths are equal.

For the triangular distribution,

$$F(x)=\int_{-1}^{x} 1-|t|\,dt, \qquad \forall x \in (-1,1).$$

If we assume that a minimizing search strategy exists which has only finitely many turning points, denote these as $\pm x_1, \mp x_2, \ldots, -1, 1$, where there are n turning points, all written as positive numbers with signs attached. Since the distribution is symmetric about 0, we can choose the signs so that $x_n = 1$ and $-x_{n-1} = -1$. Then we have

$$\Delta(\{x_j\})=\sum_{j=1}^{n} x_j(1-|F(x_j)-F(x_{j-1})|)$$
$$=\sum_{j=1}^{n} x_j(\tfrac{1}{2}(1-x_j)^2+\tfrac{1}{2}(1-x_{j-1})^2).$$

If we define a sequence $\{y_j\}$ by $y_j = x_j$, $\forall 1 \leqq j \leqq n-2$, $y_{n-1} = t$, and $y_n = y_{n+1} = 1$, then the search strategy obtained from the y's by alternating signs gives

$$\Delta(\{y_j\})=\sum_{j=1}^{n} y_j(\tfrac{1}{2}(1-y_j)^2+\tfrac{1}{2}(1-y_{j-1})^2),$$

and

$$\Delta(\{x_j\})-\Delta(\{y_j\})=(1-y_{n-1})(\tfrac{1}{2}(1-x_{n-2})^2)-(y_{n-1}+1)(\tfrac{1}{2}(1-y_{n-1})^2)$$
$$=(1-t)(\tfrac{1}{2}(1-x_{n-2})^2-\tfrac{1}{2}(t+1)(1-t))$$
$$>0 \qquad \text{iff } t^2>1-(1-x_{n-2})^2.$$

Thus, every search strategy with finitely many turning points is not minimal.

If $\{x_n\}$ is a minimizing sequence, then for every n, the derivative

$$\frac{\partial}{\partial x_n}(\Delta(\{x_n\}, F)) = 0.$$

In this case, The condition gives

$$(x_n + x_{n+1})(1 - x_n) - \tfrac{1}{2}(1 - x_n)^2 - \tfrac{1}{2}(1 - x_{n-1})^2 = 0.$$

Numerical calculation (to 16 places) then gives $x_1 = .656092985176458$, $x_2 = .9697418698558101$, $x_3 = .9997710916731093$, and $x_4 = .9999999602635702$. For $n > 4$, $x_n = 1$ to 16 places. More accurately, x_n is approximately $1 - \frac{1}{4}(1 - x_{n-1})^2$, $\forall n > 4$.

3. Symmetric distributions

3.1. Definition. A probability distribution F is called *symmetric* if $F(x) + F(-x) = 1$, $\forall x \in R$.

For symmetric distributions we define $G(x) = 1 - F(x)$, $\forall x > 0$. For such distributions, the search strategy $x_1, -x_2, x_3, -x_4, \ldots$ is equivalent to $-x_1, x_2, -x_3, x_4, \ldots$. Thus, we will use the sequence $\{x_n\}$ of positive terms to stand for either of these search strategies and adjust our formulae accordingly. We then have

$$0 < x_1 < x_3 < \cdots < x_{2j+1} < \cdots \quad \text{and} \quad 0 < x_2 < x_4 < \cdots < x_{2j} < \cdots.$$

However,

3.2. Lemma. *If $\{x_n\}$ is a minimizing search strategy for a symmetric distribution F, then for all turning ponts satisfying $G(x_n) > 0$,*

$$0 < x_1 \leqq x_2 \leqq x_3 \leqq x_4 \leqq x_5 \leqq x_6 \leqq \cdots.$$

Proof. We proceed by contradiction. Assume $0 < x_1 \leqq \cdots \leqq x_k < x_{k+2} < x_{k+1}$. Choose $r \geqq 0$ so that r is even and $x_{k+r} \leqq x_{k+1} < x_{k+r+2}$. Define $y_n = x_n$, $\forall 1 \leqq n \leqq k$, $y_{k+1} = x_{k+2}$, $y_{k+2} = x_{k+1}$, $y_{k+2+j} = x_{k+r+j}$, $\forall j = 1, 2, \ldots$. Then

$$\begin{aligned}\Delta(\{x_n\}) = &\left[\sum_{j=1}^{k} x_j(G(x_j) + G(x_{j-1}))\right] + x_{k+1}(G(x_{k+1}) + G(x_k)) \\ &+ x_{k+2}(G(x_{k+2}) + G(x_{k+1})) \\ &+ \left[\sum_{j=k+3}^{k+r} x_j(G(x_j) + G(x_{j-1}))\right] + x_{k+r+1}(G(x_{k+r+1}) + G(x_{k+r})) \\ &+ \sum_{j=k+r+2}^{\infty} x_j(G(x_j) + G(x_{j-1})),\end{aligned}$$

$$\Delta(\{y_n\}) = \left[\sum_{j=1}^{k} y_j(G(y_j)+G(y_{j-1}))\right] + y_{k+1}(G(y_{k+1})+G(y_k))$$

$$+ y_{k+2}(G(y_{k+2})+G(y_{k+1}))$$

$$+ y_{k+3}(G(y_{k+3})+G(y_{k+2})) + \sum_{j=k+4}^{\infty} y_j(G(y_j)+G(y_{j-1}))$$

$$= \left[\sum_{j=1}^{k} x_j(G(x_j)+G(x_{j-1}))\right] + x_{k+2}(G(x_{k+2})+G(x_k))$$

$$+ x_{k+1}(G(x_{k+1})+G(x_{k+2}))$$

$$+ x_{k+r+1}(G(x_{k+r+1})+G(x_{k+1})) + \sum_{j=k+r+2}^{\infty} x_j(G(x_j)+G(x_{j-1})).$$

Thus,

$$\Delta(\{x_n\}) - \Delta(\{y_n\}) = x_{k+1}(G(x_{k+1})+G(x_k)) + x_{k+2}(G(x_{k+2})+G(x_{k+1}))$$

$$+ \left[\sum_{j=k+3}^{k+r} x_j(G(x_j)+G(x_{j-1}))\right] + x_{k+r+1}(G_{k+r+1}) + G(x_{x+r})$$

$$- x_{k+2}(G(x_{k+2})+G(x_k)) - x_{k+1}(G(x_{k+1})+G(x_{k+2})) -$$

$$- x_{k+r+1}(G(x_{k+r+1})+G(x_{k+1}))$$

$$= x_{k+1}(G(x_k)-G(x_{k+2})) + x_{k+2}(G(x_{k+1})-G(x_k))$$

$$+ \left[\sum_{j=k+3}^{k+r} x_j(G(x_j)+G(x_{j-1}))\right] + x_{k+r+1}(G(x_{k+r})-G(x_{k+1}))$$

$$= x_{k+2}(G(x_{k+1})-G(x_{k+2})) + (x_{k+1}-x_{k+2})(G(x_k)-G(x_{k+2}))$$

$$+ \left[\sum_{j=k+3}^{k+r} x_j(G(x_j)+G(x_{j-1}))\right] + x_{k+r+1}(G(x_{k+r})-G(x_{k+1}))$$

$$> 0,$$

since all the terms are nonnegative and the first two are positive. This contradicts the minimality of $\{x_n\}$. QED

3.3. Corollary. *On the same hypotheses,* $0 < x_1 < x_2 < \cdots$.

Proof. Assume $0 < x_1 < \cdots < x_k = x_{k+1}$, where $G(x_k) > 0$. Then define $\{y_n\}$ by $y_n = x_n$, $\forall 1 \leqq n \leqq k$, $y_n = x_{n+1}$, $\forall n > k$. We have

$$\begin{aligned}\Delta(\{x_n\})-\Delta(\{y_n\}) &= x_k(G(x_k)+G(x_{k-1}))+x_{k+1}(G(x_{k+1})+G(x_k)) \\ &\quad + x_{k+2}(G(x_{k+2})+G(x_{k+1})) \\ &\quad - y_k(G(y_k)+G(y_{k-1}))-y_{k+1}(G(y_{k+1})-G(y_k)) \\ &= x_{k+1}(G(x_{k+1})+G(x_k))>0,\end{aligned}$$

contradicting the minimality of $\{x_n\}$. QED

4. Search with normal distribution

The next theorems concern a search carried out with respect to the normal distribution Φ, where

$$\Phi(x)=\int_{-\infty}^{x}\phi(t)dt \quad \text{and} \quad \phi(t)=1/\sqrt{2\pi}\exp(-t^2/2).$$

The turning points are taken to be $x_1, -x_2, x_3, \ldots$. The function $G(x)=1-\Phi(x)$, $\forall x>0$. We observe then that any minimizing sequence $\{x_n\}$ must satisfy $\partial\Delta/\partial n=0$, $\forall n\in\mathbf{N}$, i.e.

$$(x_n+x_{n+1})\phi(x_n)=G(x_n)+G(x_{n-1}).$$

4.1. Lemma. *Let $\{x_n\}$ be a sequence of positive numbers with $\{x_n\}$ a minimizing sequence for the normal distribution. If $x_n \geqq \sqrt{3}$, then $x_{n+1}<x_n^5$.*

Proof. Suppose $x_k\geqq\sqrt{3}$ and $x_{k+1}\geqq x_k^5$. Define a sequence $\{y_n\}$ of positive numbers by

$$y_n=x_n, \quad \forall n=1,\ldots,k,$$

$$y_{k+j}=x_k+j, \qquad \forall j=1,2,\ldots.$$

We will show $A_x=\Delta(\{x_n\})>\Delta(\{y_n\})=A_y$, contradicting the minimality in the hypothesis. Abbreviate x_k as x, and note that

$$\Delta(\{x_n\})-\Delta(\{y_n\})=B_x-B_y,$$

where

$$\begin{aligned}B_x=\sum_{n=k+1}^{\infty}x_n(G(x_n)+G(x_{n-1}))&>x_{k+1}(G(x)+G(x_{k+1})) \\ &\geqq x^5G(x) \\ &>x^5\left(\frac{1}{x}-\frac{1}{x^3}\right)\phi(x) \\ &=x^4\left(1-\frac{1}{x^2}\right)\phi(x) \\ &\geqq 9\cdot\tfrac{2}{3}\phi(x)=6\phi(x),\end{aligned}$$

and

$$\begin{aligned}
B_y &= \sum_{n=k+1}^{\infty} y_n(G(y_n)+G(y_{n-1}))\\
&= \sum_{j=1}^{\infty}(x+j)(G(x+j)+G(x+j-1))\\
&= (x+1)G(x)+\sum_{j=1}^{\infty}(x+j+x+j+1)G(x+j)\\
&< (x+1)\frac{1}{x}\phi(x)+\sum_{j=1}^{\infty}\frac{2x+2j+1}{x+j}\phi(x+j)\\
&= \phi(x)\left(1+\frac{1}{x}+\sum_{j=1}^{\infty}\left(2+\frac{1}{x+j}\right)\exp\left[-\tfrac{1}{2}(2xj+j^2)\right]\right)\\
&= \phi(x)\left(1+\frac{1}{x}+\sum_{j=1}^{\infty}\left(2+\frac{1}{x+j}\right)e^{-j^2/2}e^{-jx}\right).
\end{aligned}$$

Since

$$\left(2+\frac{1}{x+j}\right)e^{-j^2/2}\leqq\left(2+\frac{1}{1+\sqrt{3}}\right)e^{-1/2},\quad \forall j,$$

$$B(y)\leqq\phi(x)\left(1+\frac{1}{\sqrt{3}}+\left(2+\frac{1}{1+\sqrt{3}}\right)e^{-1/2}\frac{e^{-\sqrt{3}}}{1-e^{-\sqrt{3}}}\right)<5\phi(x).$$

Thus, $\Delta(\{x_n\})-\Delta(\{y\})=B_x-B_y>\phi(x)>0$, contradicting the minimality of $\{x_n\}$. QED

4.2. Lemma. *For $x>0$, $G(x)/\phi(x)$ is a decreasing function.*

Proof. The derivative is

$$\frac{x}{\phi(x)}\left(G(x)-\frac{1}{x}\phi(x)\right)<0.$$

4.3. Lemma. *For $x_n\geqq\sqrt{3}$,*

$$x_n-x_{n-1}<\frac{2}{x_n}\left(\ln\left(x_n^6+x_n^4+2x_n^2+2-\frac{2}{x_n^2-1}\right)\right).$$

PROOF. Since $x_n \geqq \sqrt{3}$, $x_{n+1} < x_n^5$. Thus

$$\begin{aligned} x_n^5 + x_n &> (G(x_n) + G(x_{n-1}))/\phi(x_n) \\ &> G(x_{n-1})/\phi(x_n) \\ &= \frac{G(x_{n-1})}{\phi(x_{n-1})}\frac{\phi(x_{n-1})}{\phi(x_n)} \\ &> \frac{G(x_n)}{\phi(x_n)}\exp[-(x_{n-1}^2 - x_n^2)/2] \\ &> \left(\frac{1}{x_n} - \frac{1}{x_n^3}\right)\exp[(x_n + x_{n-1})(x_n - x_{n-1})/2]. \end{aligned}$$

Thus

$$x_n^5 + x_n > \frac{x_n^2 - 1}{x_n^3}\exp[\tfrac{1}{2}(x_n)(x_n - x_{n-1})],$$

$$\frac{x_n^8 + x_n^4}{x_n^2 - 1} > \exp[\tfrac{1}{2}x_n(x_n - x_{n-1})],$$

$$\ln\left(x_n^6 + x_n^4 + 2x_n^2 + 2 - \frac{2}{x_n^2 - 1}\right) > \frac{1}{2}x_n(x_n - x_{n-1}).$$ QED

4.4. COROLLARY. *If $\{x_n\}$ is a minimizing search plan for the normal distribution, then*

$$x_n - x_{n-1} < 4.1466, \quad \forall n.$$

PROOF. If $x_n \geqq 4.1466$, *then*

$$\frac{2}{x_n}\ln\left(x_n^6 + x_n^4 + 2x_n^2 + 2 + \frac{2}{x_n^2 - 1}\right) < 4.1466.$$

If $x_n < 4.1466$, then so is $x_n - x_{n-1}$. QED

4.5. LEMMA.

$$x_n - x_{n-1} < \left(\frac{2}{x_n}\right)\ln(2x_n^4 - 4.1466x_n^3)/(x_n^2 - 1), \qquad \forall n.$$

PROOF. Same as Lemma 4.2, except using Corollary 4.4 to show $x_n + x_{n+1} < 2x_n + 4.1466$. QED

4.6. COROLLARY. $x_n - x_{n-1} < 2.6$.

PROOF. Same as Corollary 4.4.

4.7. COROLLARY. $x_n - x_{n-1} < 2.5$.

PROOF. Same as Lemma 4.5 and Corollary 4.6, using 2.6 instead of 4.1466.

QED

4.8. COROLLARY. *On the same hypotheses,*

$$x_n - x_{n-1} < (2/x_n)\ln((2x_n^4 - 2.5x_n^3)/(x_n^2 - 1)), \qquad \forall n.$$

PROOF. Same.

4.9. LEMMA. *On the same hypotheses,*

$$x_n - x_{n-1} > \frac{\ln(2x_n - 2.5x_n - 1 + 2.5/x_n)}{x_n}.$$

PROOF.

$$2x_n < x_n + x_{n+1} = G(x_n)/\phi(x_n) + \frac{G(x_{n-1})}{\phi(x_n)}$$

$$< \frac{1}{x_n} + \frac{G(x_{n-1})}{\phi(x_{n-1})}\frac{\phi(x_{n-1})}{\phi(x_n)}$$

$$< \frac{1}{x_n} + \frac{1}{x_{n-1}}\exp[-\tfrac{1}{2}(x_{n-1}^2 - x_n^2)]$$

$$< \frac{1}{x_n} + \frac{1}{x_n - 2.5}\exp[\tfrac{1}{2}(x_n + x_{n-1})(x_n - x_{n-1})];$$

but

$$\left(2x_n - \frac{1}{x_n}\right)(x_n - 2.5) < \exp[x_n(x_n - x_{n-1})].$$

Thus

$$x_n - x_{n-1} > \frac{1}{x_n}\ln\left(2x_n^2 - 2.5x_n - 1 + \frac{2.5}{x_n}\right). \qquad \text{QED}$$

4.10. COROLLARY. *Given any sequence* $\{x_n\}$ *satisfying*

$$x_{n+1} \geqq x_n \quad \textit{and} \quad (x_n + x_{n+1})\phi(x_n) = G(x_n) + G(x_{n-1}), \qquad \forall n \in \mathbf{N},$$

we have $x_n/\sqrt{n} \to \infty$.

PROOF. For each n, $x_n - x_{n-1} < 2.5$ implies

$$x_n - x_{n-1} > \frac{1}{x_n}\ln\left(2x_n^2 - 2.5x_n - 1 + \frac{2.5}{x_n}\right) > \frac{\ln 2 + 2\ln x_n}{x_n}, \qquad \forall x_n \geqq 2,$$

as shown in Lemma 4.8. Thus, x_n grows faster than $\sqrt{n}$. QED

4.11. COROLLARY. *On the same hypothesis,*

$$x_n = o(n^{1/2+\varepsilon}), \qquad \forall \varepsilon > 0.$$

PROOF. Wlog $\varepsilon < \frac{1}{2}$. $x_n - x_{n-1} < x_n^{-1+2\varepsilon}$, for all x_n large enough. Therefore

$$\frac{1}{2-2\varepsilon} x_n^{2-\varepsilon} < n + C \qquad \text{for some } C \in \mathbf{R}, \quad \text{for all } n \text{ large enough,}$$

i.e.

$$x_n = O(n^{1/(2-\varepsilon)}) = o(n^{1/2+\varepsilon}). \qquad \text{QED}$$

Using these estimates, we can now prove

4.12. THEOREM. *If $\{x_n\}$ is a minimizing sequence for the linear search under the normal distribution, then*

$$\lim x_n / \sqrt{2n \ln n} = 1.$$

PROOF. We start again with the equation $(x_n + x_{n+1})\phi(x_n) = G(x_n) + G(x_{n-1})$, which we rewrite as

$$\frac{\phi(x_{n-1})}{G(x_{n-1})}\left(x_n + x_{n+1} - \frac{G(x_n)}{\phi(x_n)}\right) = \exp\left[\tfrac{1}{2}(x_n^2 - x_{n-1}^2)\right].$$

We note that $\phi(x_{n-1})G(x_n) < G(x_{n-1})\phi(x_n)$, so the left-hand side differs from

$$\frac{\phi(x_{n-1})}{G(x_{n-1})}(x_n + x_{n+1})$$

by less than 1.

Using a very coarse estimate ($x_{n-1} > \sqrt{2}, x_n > 2.53, x_n - x_{n-1} < 2.5, \forall n$), we have

$$x_{n-1} < \phi(x_{n-1})/G(x_{n-1}) < x_{n-1} + (1/x_{n-1}) + (1/x_{n-1})^3) + \cdots < x_{n-1} + 2/x_{n-1}.$$

Thus, the left-hand side lies between $x_{n-1}(x_n + x_{n+1})$ and $(x_{n-1} + 2/x_{n-1}) \times (x_n + x_{n+1}) + 1$, and thus differs from $2x_n^2$ by no more than $2.5x_n + 6.25 + 7.5/x_{n-1}$. Thus, the logarithm of the left-hand side differs from $\ln(2x_n^2)$ by no more than

$$(2.5x_n + 6.25 + 7.5/x_{n-1})/2x_n^2 = 1.25/x_n + 12.5/x_n^2 + 7.5/x_n^2 \cdot x_{n-1}$$
$$< (1.25 + 2 + 22)/x_n < 10 \quad \text{(Corollary 4.10).}$$

We now define $g_n = x_n^2/(n \ln n)$, $\forall n$, and observe that $g_n \ln n = x_n^2/n \to \infty$ and

$$\begin{aligned} x_n^2 - x_{n-1}^2 &= g_n \cdot n \ln n - g_{n-1}(n-1)\ln(n-1) \\ &= (n-(n-1))g_n \ln n + (\ln n - \ln(n-1))(n-1)g_n \\ &\quad + (n-1)\ln(n-1)(g_n - g_{n-1}), \end{aligned}$$

which differs from $g_n \ln n + g_n + (n-1)\ln(n-1)(g_n - g_{n-1})$ by less than g_n/n. Dividing by $(n-1)\ln(n-1)/2$ and substituting in the left-hand side, we have

$$\left| \frac{2\ln(2g_n n \ln n)}{(n-1)\ln(n-1)} - g_n \frac{\ln n}{\ln(n-1)} \cdot \frac{1}{n-1} - \frac{g_n}{(n-1)\ln(n-1)} - (g_n - g_{n-1}) \right| < \frac{20}{(n-1)\ln(n-1)}.$$

In other words

$$A + \frac{2\ln 2}{(n-1)\ln(n-1)} + \frac{2\ln(g_n \ln n)}{(n-1)\ln(n-1)} + \frac{2}{n-1}\frac{\ln n}{\ln(n-1)} - \frac{g_n}{n-1}\frac{\ln n}{\ln(n-1)} - \frac{g_n}{(n-1)\ln(n-1)} = g_n - g_{n-1},$$

where $|A| < 20/(n-1)\ln(n-1)$. Thus if $n > e^{24/\varepsilon}$ is large enough to assure that $g_n \ln n > 1$, and $g_n < 2-\varepsilon$, we have $2\ln 2 + 2\ln g_n < 4\ln 2 < 1.3$, and

$$\frac{\varepsilon}{n-1} + B < \frac{2-g_n}{n-1}\frac{\ln n}{\ln(n-1)} + \frac{2\ln(g_n \ln n)}{(n-1)\ln(n-1)} + B < g_n - g_{n-1},$$

where $B = A + (2\ln 2 - g_n)/(n-1)\ln(n-1)$, so that $|B| < 22/(n-1)\ln(n-1)$, and $g_n - g_{n-1} > (\varepsilon - 12/\ln(n-1))/(n-1) > \varepsilon/(2(n-1))$.

Since the same inequality holds for all successive n until $g_{n+k} > 2-\varepsilon$, it follows that $g_n > 2-\varepsilon$ infinitely often. On the other hand, the same restrictions on n assure that if $g_{n-1} > 2-\varepsilon$, then $g_n > 2-\varepsilon$ since $g_n - g_{n-1} > 0$ if $g_n < 2-\varepsilon$. Thus $g_n > 2-\varepsilon$ for all n large enough.

On the other side, if n is so large that $\ln\ln n < (\varepsilon \ln n)/8$, $\ln g_n < \varepsilon \ln n/8$ (by Corollary 4.11; see below), and $g_n > 2+\varepsilon$, we have

$$\begin{aligned} &C - \frac{\varepsilon}{n-1} + \frac{2\ln g_n}{(n-1)\ln(n-1)} + \frac{2\ln\ln n}{(n-1)\ln(n-1)} \\ &> C + \frac{(2-g_n)}{(n-1)}\frac{\ln(n)}{\ln(n-1)} + \frac{2\ln(g_n \ln n)}{(n-1)\ln(n-1)} - \frac{g_n}{(n-1)\ln(n-1)} \\ &= g_n - g_{n-1}, \end{aligned}$$

where $C = A + 2\ln 2/(n-1)\ln(n-1)$, so that $|C| < 21/(n-1)\ln(n-1)$. By Corollary 4.11, $g_n/n^\eta \to 0$, $\forall \eta > 0$, so that $\ln g_n/\ln n \to 0$. Thus, for n large enough so that $\ln g_n/\ln n < \varepsilon/8$, we have

$$\begin{aligned} C - \frac{\varepsilon}{2(n-1)} &> C - \frac{\varepsilon}{n-1} + \frac{\varepsilon}{4(n-1)} + \frac{2\ln\ln n}{(n-1)\ln(n-1)} \\ &> C - \frac{\varepsilon}{n-1} + \frac{2\ln g_n}{(n-1)\ln n} + \frac{2\ln\ln n}{(n-1)(\ln(n-1))} \\ &> C - \frac{\varepsilon}{n-1} + \frac{2\ln g_n}{(n-1)\ln(n-1)} + \frac{2\ln\ln n}{(n-1)\ln(n-1)} \\ &> g_n - g_{n-1}. \end{aligned}$$

Thus, as before, g_{n+k} becomes and remains less than $2+\varepsilon$ as $k \to \infty$. QED

5. Calculation of turning points for the normal search

Given the properties of the turning points of the minimal search procedure for the normal distribution, we might seek to find sequences satisfying the basic recursion relation

$$(x_n + x_{n+1})\phi(x_n) = G(x_n) + G(x_{n-1}),$$

where $G(x) = 1 - \Phi(x)$, $\forall x > 0$. Among those, we will identify the minimal value of $\Delta(\{x_n\})$. Such a program involves the assignment of a value to x_1, and taking $x_0 = 0$, to generate a sequence $\{x_n\}$ using the recursion equation [6]. It quickly transpires that the method is highly unstable. For values smaller than the correct x_1, we do not have $x_n > x_{n-1}$, $\forall n$, and for values greater than the correct one, the estimate on $x_n - x_{n-1}$ is soon exceeded. This is because any particle of error is quickly magnified. For any change in x_1, the change in x_2 can be seen by calculating $\partial x_2/\partial x_1$ from the relation

$$x_2 = G(x_1)/\phi(x_1) + G(0)/\phi(x_1) - x_1.$$

This gives

$$\begin{aligned} \frac{\partial x_2}{\partial x_1} &= \frac{-\phi(x_1)^2 + x_1\phi(x_1)G(x_1)}{\phi(x_1)^2} + \frac{G(0)x_1\phi(x_1)}{(\phi(x_1))^2} - 1 \\ &= -1 + \frac{x_1(G(x_1))}{\phi(x_1)} + \frac{x_1 G(0)}{\phi(x_1)} - 1 \\ &= x_1\left(\frac{G(x_1)}{\phi(x_1)} + \frac{G(0)}{\phi(x_1)}\right) - 2 \\ &= x_1(x_1 + x_2) - 2. \end{aligned}$$

Writing d_n for $\partial x_n/\partial x_1$, we obtain in general

$$d_{n+1}=\frac{\partial x_{n+1}}{\partial x_1}=\left[-\frac{\phi(x_n)^2+x_n\phi(x_n)G(x_n)}{\phi(x_n)^2}+\frac{G(x_{n-1})x_n\gamma(x_n)}{\phi(x_n)^2}-1\right]d_n-\frac{\phi(x_{n-1})}{\phi(x_n)}d_{n-1}$$

$$=[x_n(x_n+x_{n+1})-2]d_n-[\phi(x_{n-1})/\phi(x_n)]d_{n-1}.$$

As we shall see below, these numbers are truly spectacular.

For numbers less than $x_1=1.440854109$, we get $x_n<x_{n-1}$, $\exists n\leqq 8$. Taking that value of x_1, an Apple computer working on Basic yields the following values for the x_n:

$$x_2=2.62758012\quad x_3=3.63220007\ .\ x_4=4.52034106\quad x_5=5.32662513$$
$$x_6=6.06835073\quad x_7=6.52933041\quad x_8=-3.44945888,$$

with garbage as output thereafter. However, increasing x_1 by 10^{-30} gives the same value for x_2, increases x_3 by $7\cdot 10^{-8}$, x_4 by $162\cdot 10^{-8}$, increases x_5 by $6634\cdot 10^{-8}$, x_6 by more than .0038, and x_7 by nearly 3. It gives x_8 as more than 12, and then quickly overflows the computer. For larger values of x_1, the overflow comes sooner, and all the values are larger.

Given the above values of x_n, the values of d_n are approximately

$$d_2=3.9,\quad d_3=70,\quad d_4=2047,\quad d_5=9000,$$
$$d_6=5.5\cdot 10^6,\quad d_7=43\cdot 10^8,\quad d_8=4.75\cdot 10^{10}$$

Thus, we can understand the instability in the data, even on theoretical grounds, and to these we must add the foibles of the machinery. Since d_n continues to grow, and in fact

$$d_{n+1}>(x_n(x_n+x_{n+1})-2)(d_n-d_{n-1})+((x_n-x_{n-1})(x_n+x_{n+1})-2)d_{n-1},$$

thus we see that there is only one value for which our derived inequalities (section 4) are valid.

However, these data are only roughly correct, due to difficulties in the calculational systems. To obtain finer extimates, we extrapolate approximate values of x_n for n between 8 and 21, and then we use the recurrence relation not to extrapolate x_{n+1} from x_n and x_{n-1}, but to fine-tune x_n from the putative values of x_{n-1} and x_{n+1}. Repeated re-adjustments finally give stable readings. The first four, which agree with the values in [6] up to round-off error, are

$$1.44085411,\qquad 2.62758012,\qquad 3.63220012,\qquad 4.52034243.$$

The next four, which differ from the values in [6] by amounts increasing from 10^{-6} to .000336, are

5.32668134, 6.07157768, 6.76811167, 7.42526253.

Our values for the next twelve turning points are

8.04950858,	8.64570659,	9.21761012,	9.76819253,
10.2998576,	10.8145824,	11.3140164,	11.7995539,
12.2723858,	12.7335393,	13.1839076,	13.6241273.

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